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On the gauge dependence of spectral functions†

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Abstract. An integral relation is derived between charged source spectral functions in photon gauges differing by longitudinal terms $ak_\mu k_\nu/k^4$. The a dependences of the spectral and Green functions supplied by the gauge technique automatically satisfy these integral relations.

1. Introduction

The relations between Green functions in different Lorentz covariant gauges were obtained many years ago (Landau and Khalatnikov 1956, Fradkin 1956) and subsequently rederived by functional (Zumino 1960, Bialynicki–Birula 1960) and other (Okubo 1960) methods. These relations serve to connect the charged particle Green functions for photon propagators $D_{\mu\nu}(x)$ differing by the longitudinal term $\partial_\mu\partial_\nu M$ and they were effectively used by Johnson and Zumino (1959) and Zumino (1960) to highlight the gauge dependence of the renormalisation constants as well as the infrared and ultraviolet behaviours of the electron propagator. The connection between the Green functions involves the phase factor $\exp(\mathrm{i}e^2M(x))$, and, as such, is most easily expressed in configuration space. In this paper we wish to discuss the gauge dependence of the charged propagator spectral function ρ; since this is couched in momentum space the relationship cannot be expected to remain so simple. For the class of covariant gauges where $D_{\mu\nu}(k)$ varies by $ak_\mu k_\nu/k^4$,

$$\exp[\mathrm{i}e^2M(x)] = (-m^2x^2)^{-e^2a/16\pi^2}$$

represents the multiplicative effect of the phase factor. We shall determine (see equations (22) and (23)) the connection between ρ in two different a gauges and then demonstrate how the explicit ρ obtained (Delbourgo and West 1977a, b, Delbourgo 1977) by means of the gauge technique (Salam 1963, Delbourgo 1979) automatically satisfy the identity. To expose this more transparently, we go on to find the corresponding x space charged particle Green functions.

A rapid derivation of the gauge relations is provided in § 2, as well as the corresponding steps for the non–abelian case and why they are not particularly fruitful. In § 3 we obtain the integral connection between the spectral functions and finally we study the repercussions for the gauge technique in § 4.

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2. Gauge dependence of Green functions

Here we shall rapidly derive the gauge properties of amplitudes by functional methods. We shall treat the abelian case first and stick to Zumino's (1960) notation. Further on, we shall discuss the generalisation to the non-abelian case and show that it fails to provide closed form relations that are of any use.

In a gauge† $a_\mu A^\mu = A$ one begins with the vacuum functionals

$$Z_\Lambda[u, v, J] = \int [dA_\mu \, d\psi \, d\bar{\psi} \, dB]\psi(u_1)\ldots\psi(u_m)\bar{\psi}(v_1)\ldots\bar{\psi}(v_n) \times \exp\left[i\int(\mathscr{L} - JA - B(\Lambda - aA))\right] \tag{1}$$

where B acts as an auxiliary multiplier field, enforcing the gauge condition. As a further extension, which for the Landau gauge amounts to an addition $\partial_\mu\partial_\nu M(x-y)$ to the photon propagator $D_{\mu\nu}(x-y)$, one may envisage

$$Z_{\Lambda,M}[u, v, J] = \int dA_\mu \ldots dB \, \pi\psi(u_i)\pi\bar{\psi}(v_i) \exp\left(i\int(\mathscr{L} - JA - B(\Lambda - aA) - \tfrac{1}{2}BMB)\right] \tag{2}$$

and assume without loss that $\partial_\mu a^\mu = 1$.

The Λ, M dependence may be elucidated by gauge transforming the integration variables A, ψ, $\bar{\psi}$ by an amount depending on B:

$$A_\mu \to A_\mu - \partial_\mu\chi \qquad \psi \to \exp(ie\chi)\psi \qquad \bar{\psi} \to \exp(-ie\chi)\bar{\psi} \tag{3}$$

where

$$\chi(x) = g(x) + \int dy \, h(x-y)B(y). \tag{4}$$

We obtain

$$Z_{\Lambda,M}[u, v, J] = \int [dA_\mu \ldots dB]\pi \exp[ie(g+hB)(u_i)]\psi(u_i)\pi \exp[-ie(g+hB)(v_i)]\bar{\psi}(v_i) \times \exp\left\{i\int[\mathscr{L} - J(A-\partial)(g+hB)] - B(\Lambda - aA + g + hB) - \tfrac{1}{2}BMB\right\} \tag{5}$$

and so

$$Z_{\Lambda,M}[u, v, J] = \exp\left(-i\int(g\partial J)\right]\pi \exp(ieg(u_i))\pi \exp(-ieg(v_i))Z_{\Lambda',M'}[u, v, J] \tag{6}$$

where

$$\begin{aligned} \Lambda'(x) &= \Lambda(x) - e\Sigma h(u_i - x) + e\Sigma h(v_i - x) + \int dy \, h(y-x)(\partial J)(y) + g(x) \\ M'(x) &= M(x) + h(x) + h(-x) \end{aligned} \tag{7}$$

Clearly by suitably choosing g and h we can connect (Λ, M) to any (Λ', M').

In particular, with a little algebra, we find

$$Z_{\Lambda,M}[u, v, J] = \exp(i\theta)Z_{\Lambda,0}[u, v, J] \tag{8}$$

where

$$\theta = -\tfrac{1}{2}(\partial J) \,.\, M(\partial J) + e[\Sigma(M\partial J)(u_i) - \Sigma(M\partial J)(v_i)] + e^2(-\tfrac{1}{2}\Sigma M(u_i - u_j) + \Sigma M(u_i - v_j) - \tfrac{1}{2}\Sigma M(v_i - v_j)). \tag{9}$$

† $(a_\mu A^\mu)(x) = \int dy \, a_\mu(x-y)A^\mu(y)$

Expanding in J and suppressing Λ we find for the propagators and vertex function

$$D_{\mu\nu}(x-y;M)=D_{\mu\nu}(x-y;0)-\partial_\mu\partial_\nu M(x-y) \tag{10}$$

$$S(x-y;M)=\exp(ie^2(M(x-y)-M(0))]S(x-y;0) \tag{11}$$

$$\{D_{\mu\nu}S\Gamma^\nu S\}(x,y,z;M)=\exp[ie^2(M(x-y)-M(0))][\{D_{\mu\nu}S\Gamma^\nu S\}(x,y,z;0) + ieS(x-y;0)\partial^z_\mu\{M(x-z)-M(y-z)\}] \tag{12}$$

The above relations are for unrenormalised fields and expectation values. Since (Johnson and Zumino 1959)

$$Z_2(M)/Z_2(0)=\exp(-ie^2M(0)) \tag{13}$$

one deduces the following relations between renormalised Green functions:

$$S(x;M)=\exp(ie^2M(x))S(x;0) \tag{11'}$$

$$\{DS\Gamma S\}(x,y,z;M)=\exp(ie^2M(x-y))[\{DS\Gamma S\}(x,y,z;0) + ieS(x-y;0)\partial^z(M(x-y)-M(y-z))] \tag{12'}$$

Note that since

$$e_r^2=Z_3e_u^2, \qquad M_r=Z_3^{-1}M_u, \qquad e_r(S\Gamma SD)_r=Z_2^{-1}e_u(S\Gamma SD)_u$$

the combinations e^2M and $e\Gamma SD$ are renormalisation invariant. Indeed, consistent with the transformation properties (12), it is easily verified that the renormalised Dyson–Schwinger equation

$$Z_2^{-1}\delta(x)=(i\gamma\cdot\partial-m_0)S(x)+ie\gamma_\mu\{S\Gamma_\nu SD^{\mu\nu}\}(x,0,x)$$

is valid for any fixed value of M, as we know it must be.

Let us generalise from QED to QCD to see how the argument goes awry. We begin with the Faddeev–Popov modified version of (2),

$$Z_{\Lambda\mu}=\int[dA_\mu\ldots dB]\Delta(A)\pi\psi(u)\pi\bar\psi(v)\exp\left[i\int(\mathscr{L}-JA-B(\Lambda-aA)-\tfrac{1}{2}BMB)\right]. \tag{14}$$

The measure $[dA]\Delta(A)$ is invariant only under certain A-dependent transformations of Slavnov (1972) type,

$$\delta A_\mu=-D_\mu\delta\chi(A)=-D_\mu(a\cdot D)^{-1}\delta\Lambda \tag{15}$$

where D is the normal covariant derivative and $\delta\Lambda$ is A independent. (The usual proof of this invariance (Lee and Zinn-Justin 1973) is perhaps rather complicated, so we thought it worthwhile to present a more transparent and general derivation in Appendix A). As for QED, $\delta\Lambda$ can depend on B. We obtain

$$Z_{\Lambda\mu}=\int[dA\ldots dB]\Delta(A)\pi\exp(ieT\delta\chi)\psi(u)\,.\,\pi\exp(-ie\bar T\delta\chi)\bar\psi(v) \times\exp\left[i\int(\mathscr{L}-J(A-D\delta\chi)+B(aA-\Lambda+\delta\Lambda)-\tfrac{1}{2}BMB)\right] \tag{16}$$

as the direct analogue of (5). In (16), T is the generator in the fermion representation, and is one of the unpleasant complications over the abelian case. For one thing, the factors $\exp(ieT\delta\chi)$ and $\exp(-ie\bar T\delta\chi)$ are not phases, so cannot be combined with the rest of the expression as a change in Λ. For another, they depend in a complicated way on A, so replacement of A by $\mathrm{i}\delta/\delta J$ can only lead to relations involving infinite numbers of Green functions. In the absence of fermions the situation is hardly better due to the $\delta\chi$ term in the square brackets. The simplicity, even for infinitesimal transformations, is therefore lost in the non-abelian case, although the gauge–dependence of the scaling properties has been further investigated (Hosoya and Sato 1974, Tarasov and Vladimirov 1977).

3. Gauge dependence of spectral functions

In view of our failure with QCD we restrict ourselves to QED hereafter. Recall that when the photon propagator is varied as

$$D^M_{\mu\nu}(k) = D^0_{\mu\nu}(k) + k_\mu k_\nu M(k) \tag{10}$$

the renormalised propagators for different M values are connected by

$$S(x; M) = \exp(\mathrm{i}e^2 M(x)) S(x; 0) \tag{11'}$$

On the other hand, we know that the propagator admits the spectral decomposition

$$S(x) = \int \rho(W) S(x \mid W)\,\mathrm{d}W \tag{17}$$

where $S(x \mid W) = (\mathrm{i}\gamma\,.\,\partial + W)\Delta(x \mid W^2)$ is the free (mass W) fermion propagator. This yields a complicated relation

$$\int \rho(W; M) S(x \mid W)\,\mathrm{d}W = \exp(\mathrm{i}e^2 M(x)) \int \rho(W; 0) S(x \mid W)\,\mathrm{d}W,$$

between the corresponding spectral functions in the two gauges. It can be simplified somewhat by taking the discontinuity of the Fourier transform:

$$-\pi\rho(p; M) = \mathrm{Im} \int \mathrm{d}W \rho(W; 0) \int \mathrm{d}^4x \exp(\mathrm{i}p\,.\,x) \exp(\mathrm{i}e^2 M(x)) S(x \mid W) \tag{18}$$

This is now an integral relation between the different ρ.

To make further progress, the form of M needs to be specified. In the class of covariant gauges

$$D_{\mu\nu}(k) = [-\eta_{\mu\nu} + k_\mu k_\nu (1-a)/k^2]/k^2 \tag{19}$$

one identifies $M(k) = -a/k^4$ at a formal level. Let us use dimensional regularisation to give meaning to $\exp[\mathrm{i}e^2 M(x)]$, instead of introducing Pauli–Villars regulators as did Johnson and Zumino (1959). In $2l$ dimensions, we substitute $e^2(m^2)^{2-l}$ for e^2 to maintain the coupling constant dimensionless, m being the electron mass. Hence, in the four-dimensional limit

$$\begin{aligned} e^2 M(x) &\to -\lim_{l\to 2} e^2(m^2)^{2-l} \int \bar{d}^{2l}k \exp(-\mathrm{i}k\,.\,x)\frac{a}{k^4} \\ &= \lim_{l\to 2} \frac{e^2(m^2)^{2-l} a}{4(l-2)} \int \bar{d}^{2l}k \exp(-\mathrm{i}k\,.\,x)\frac{\partial}{\partial k^\mu}\frac{\partial}{\partial k_\mu}\frac{1}{k^2} \\ &= \lim_{l\to 2} \frac{e^2(m^2)^{2-l}}{4(2-l)} x^2 D(x) \end{aligned}$$

where $iD(x)=\Gamma(l-1)(-x^2+i0)^{1-l}/4\pi^l$ is the causal massless propagator for arbitrary l. Thus, up to an x independent constant factor,

$$e^2M(x)\to ie^2a\ln(-m^2x^2)/16\pi^2$$

or

$$\exp(ie^2M(x))=(-m^2x^2)^{-e^2a/16\pi^2}\equiv(-m^2x^2)^{-a\epsilon} \tag{20}$$

The abbreviation $\epsilon=e^2/16\pi^2$ has been used as it recurs throughout.

In order to discover the relation between $\rho(W;a)$ and $\rho(W;0)$, we shall need the transform

$$\int d^4x\,\exp(ipx)(-m^2x^2)^{-a\epsilon}(i\gamma.\partial+W)\{-iWK_1[W(-x^2)^{1/2}]/4\pi^2(-x^2)^{1/2}\}$$
$$=\int_0^\infty dr\frac{r^2J_1[r(-p^2)^{1/2}]}{(-p^2)^{1/2}}.(m^2r^2)^{-a\epsilon}W^2\left[\frac{K_1(Wr)}{r}+i\gamma.x\frac{K_2(Wr)}{r^2}\right]$$
$$=\left(\frac{4m^2}{W^2}\right)^{-a\epsilon}\frac{\Gamma(1-a\epsilon)}{W}\left[\Gamma(2-a\epsilon)F\left(1-a\epsilon,2-a\epsilon;2;\frac{p^2}{W^2}\right)\right.$$
$$\left.+\frac{\gamma.p}{2W}\Gamma(3-a\epsilon)F\left(1-a\epsilon,3-a\epsilon;3;\frac{p^2}{W^2}\right)\right]. \tag{21}$$

Since the hypergeometric solution $F(\alpha,\beta;\gamma;z)$, regular at $z=0$, has a branch point at $z=1$ with a discontinuity given by

$$\mathrm{Im}\{\Gamma(\alpha)\Gamma(\beta)F(\alpha,\beta;\gamma;z)\}$$
$$=-\frac{\pi\Gamma(\gamma)\theta(z-1)}{\Gamma(1-\alpha-\beta+\gamma)}(z-1)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-z)$$

there follows the fundamental integral relation

$$\rho(p;a)=\left(\int_m^p+\int_{-p}^{-m}\right)dW\frac{\rho(W;0)}{W}\left(\frac{W}{2m}\right)^{2a\epsilon}\frac{(p^2/W^2-1)^{-1+2a\epsilon}}{\Gamma(2a\epsilon)}$$
$$\times[F(a\epsilon,1+a\epsilon;2a\epsilon;1-p^2/W^2)$$
$$+W^{-1}\gamma.pF(a\epsilon,2+a\epsilon;2a\epsilon;1-p^2/W^2)] \tag{22}$$

between the spinor spectral functions. The connection is visibly non-trivial; nor is it more trivial in the companion scalar case,

$$\rho(p^2,a)=\int_{m^2}^{p^2}dW^2\frac{\rho(W^2;0)}{W^2}\left(\frac{W^2}{4m^2}\right)^{a\epsilon}\frac{(p^2/W^2-1)^{-1+2a\epsilon}}{\Gamma(2a\epsilon)}F(a\epsilon,1+a\epsilon;2a\epsilon;1-p^2/W^2) \tag{23}$$

after one traces out the parallel argument.

At asymptotic values the relations look more tractable. In the infrared limit, with scalars (near $p^2=m^2$), the connection reads

$$\rho(p^2;a)\simeq\int_{m^2}^{p^2}\frac{dW^2}{m^2}\rho(W^2;0)\frac{2^{-2a\epsilon}}{\Gamma(2a\epsilon)}\left(\frac{p^2}{W^2}-1\right)^{-1+2a\epsilon}$$

and with spinors (near $\gamma . p = m$) it reads, very similarly,

$$\rho(p; a) \simeq \left(1 + \frac{\gamma . p}{m}\right) \int_m^p \frac{\mathrm{d}W}{m} \rho(W, 0) \frac{2^{-2a\epsilon}}{\Gamma(2a\epsilon)} \left(\frac{p^2}{W^2} - 1\right)^{-1+2a\epsilon}.$$

If we assume that there are threshold singularities

$$m^2 \rho(p^2; 0) \sim (p^2/m^2 - 1)^{\zeta_0} \qquad \text{and} \qquad m\rho(p; 0) \sim (p/m - 1)^{\zeta_{1/2}}$$

in the Landau gauge ($a = 0$), we readily arrive at the general infrared behaviours,

$$\begin{aligned} m^2 \rho(p^2; a) &\sim (p^2/m^2 - 1)^{\zeta_0 + 2a\epsilon} \\ m\rho(p; a) &\sim (1 + \gamma . p/m)(p^2/m^2 - 1)^{\zeta_{1/2} + 2a\epsilon}. \end{aligned} \tag{24}$$

Likewise, in the ultraviolet domain ($p^2 \gg m^2$) for integrals dominated by the upper end point and the postulated asymptotic behaviour†

$$m^2 \rho(p^2; 0) \simeq (p^2/m^2)^{\eta_0} \qquad m\rho(p; 0) \simeq (p^2/m^2)^{\eta_{1/2}}$$

we arrive, via a scaling of (22) and (23), at the general ultraviolet characteristics

$$m^2 \rho(p^2; a) \simeq (p^2/m^2)^{\eta_0 + a\epsilon} \qquad m\rho(p; a) \simeq (p^2/m^2)^{\eta_{1/2} + a\epsilon}. \tag{25}$$

The connecting formulae (22) and (23) do not, however, tell us what ρ is in a chosen gauge or what values the coefficients η and ζ take. To find these out one is obliged to turn to perturbation theory and the renormalisation group. In QED we know by infrared freedom that

$$\zeta_0 = \zeta_{1/2} = -1 + 6\epsilon$$

which may be substituted in (26) to give the complete infrared answers for ρ or the Green functions. At the other, high-energy extreme, if we suppose that the propagators‡ are indeed governed by a dimension η as in (25), then up to *first order in* e^2 a simple calculation based upon extraction of leading logarithms gives

$$\eta_0 = \eta_{1/2} = -1 - 3\epsilon$$

From (25) one can read off the ultraviolet behaviour in any covariant gauge a. The virtues of the Landau and Yennie gauges were emphasised long ago by Johnson and Zumino (1959).

4. The gauge technique

If we knew $\rho(W)$ for all W in a given a gauge, we could determine ρ and S in any other a gauge fairly easily. Unfortunately we do not know the full ρ (for, if we did, it would mean that we should have solved QED completely) except in some approximation. One such approximation is provided by the gauge technique which offers, at a first level, the

† Strictly, we are considering the even part of $\rho(p)$, viz. $\rho(p) + \rho(-p)$, as this supplies the dominant behaviour (Atkinson and Slim 1979) of S at asymptotic values.

‡ In so far as the propagators have the same $p \to \infty$ (and $x \to 0$) behaviour as their spectral functions, the small distance limit is more easily discussed directly in configuration space from (11) and (20).

Landau gauge spectral functions†

$$m^2\rho(W^2;0)=\frac{(W^2/m^2-1)^{-1-6\epsilon}}{2^{-6\epsilon}\Gamma(-6\epsilon)}F(-3\epsilon,1-3\epsilon;-6\epsilon;1-W^2/m^2) \quad (26)$$

$$m\rho(W;0)=\frac{\epsilon(W)(W^2/m^2-1)^{-1-6\epsilon}}{2^{-6\epsilon}\Gamma(-6\epsilon)} \times\left[\frac{m}{W}F(-3\epsilon,-3\epsilon;-6\epsilon;1-W^2/m^2) +F(-3\epsilon,1-3\epsilon;-6\epsilon;1-W^2/m^2)\right] \quad (27)$$

These can be substituted in (24) and (25) respectively to give us the ρ for $a\neq 0$. In the scalar case one is confronted by the integral

$$\int_1^x dy\frac{y^{-1+a\epsilon}}{2^{(2a-6)\epsilon}}\frac{(x/y-1)^{-1+2a\epsilon}}{\Gamma(2a\epsilon)}\frac{(y-1)^{-1-6\epsilon}}{\Gamma(-6\epsilon)}F(a\epsilon,1+a\epsilon;2a\epsilon;1-x/y) \times F(-3\epsilon,1-3\epsilon;-6\epsilon;1-y)$$

wherein $x=p^2/m^2$ and $y\equiv W^2/m^2$. This is evaluated in Appendix B and the result is

$$m^2\rho(W^2;a)=\frac{(W^2/m^2-1)^{-1+(2a-6)\epsilon}}{2^{(2a-6)\epsilon}\Gamma((2a-6)\epsilon)}F((a-3)\epsilon,1+(a-3)\epsilon;(2a-6)\epsilon;1-W^2/m^2)) \quad (28)$$

which is precisely the answer supplied the gauge technique for any a. (In the spinor case the integral cannot be reduced to a simple ${}_2F_1$ function‡, and we shall not quote it.) We have thus verified that the gauge techniques ρ are entirely consistent with the gauge dependence (23) expected on general grounds; this is reassuring if not especially surprising bearing in mind the gauge covariance of the technique.

We can clear much of the mystery and avoid these contortions with hypergeometric functions if we go to x space. As an intermediate step, evaluate

$$\begin{aligned}\Delta(p;0)&=\int_{m^2}^{\infty}dW^2\,\rho(W^2;0)/(p^2-W^2)\\ &=\int_0^{\infty}\frac{x^{-1-6\epsilon}(x+1-p^2/m^2)^{-1}}{2^{-6\epsilon}\Gamma(-6\epsilon)m^2}F(-3\epsilon,1-3\epsilon;-6\epsilon;-x)\\ &=-m^{-2}2^{6\epsilon}\Gamma(1+3\epsilon)\Gamma(2+3\epsilon)F(1+3\epsilon,2+3\epsilon;2;p^2/m^2).\end{aligned} \quad (29)$$

Thus

$$\begin{aligned}i\Delta(x;0)&=\int_0^{\infty}dq\,q^2J_1[q(-x^2)^{1/2}]\Delta((-q^2)^{1/2};0)/4\pi^2(-x^2)^{1/2}\\ &=mK_1[m(-x^2)]^{1/2}(-m^2x^2)^{3\epsilon}/4\pi^2(-x^2)^{1/2}\end{aligned} \quad (30)$$

† The normalisation factors, which are not provided by the technique, have been carefully introduced to give the correct free propagators as $\epsilon\to 0$ and to maintain their simple character in any gauge. Thus $\rho\to\delta(W^2-m^2)$ or $\delta(W-m)$ as the coupling vanishes.

‡ Rather it leads to ${}_4F_3$ functions, again in conformity with the differential equation for $\rho(W;a)$ provided by the technique.

in the Landau gauge, according to the initial gauge approximation. Hence in other gauges,

$$i\Delta(x;a)=(-m^2x^2)^{(3-a)\epsilon}mK_1[m(-x^2)^{1/2}]/4\pi^2(-x^2)^{1/2}. \tag{31}$$

The Yennie gauge $a=3$ would thus have the free propagator as the starting function.

Carrying out the same steps in the spinor case, we find (see also Khare and Kumar 1978)

$$S(p;0)=-2^{6\epsilon}\Gamma^2(1+3\epsilon)[(\gamma.p)^{-1}(F(1+3\epsilon,1+3\epsilon;1;p^2/m^2)-1)$$
$$+m^{-1}(1+3\epsilon)F(1+3\epsilon,2+3\epsilon;2;p^2/m^2)] \tag{32}$$

and therefore † ($r^2\equiv -x^2$)

$$iS(x;0)=\frac{(m^2r^2)^{3\epsilon}m^2}{4\pi^2}\left(\frac{K_1(mr)}{r}+\frac{i\gamma.x}{r^2}\left\{K_2(mr)+\frac{2K_1(mr)}{mr}[\exp(-3i\pi\epsilon)S_{1+6\epsilon,0}(imr)-1]\right.\right.$$
$$\left.\left.+\frac{12i\epsilon}{m^2r^2}K_0(mr)\exp(-3i\epsilon)S_{6\epsilon,1}(imr)\right\}\right) \tag{33}$$

where $S_{\mu,\nu}$ is the Lommel function. In the limit as $\epsilon\to 0$, because $S_{1,0}(z)=1$, one verifies that (33), like (32), tends to the free-field propagator—a useful boundary condition to test the correctness of the expressions. Multiplication of (33) by $(m^2r^2)^{-a\epsilon}$ then provides the Green function for arbitrary a. We see that, unlike the scalar case, the spinor result is non-trivial no matter what a value is chosen. The reason why the scalar propagator (31) looks so simple for $a=3$ is because the e^2 correction to the scalar spectral function happens to vanish for that gauge; this never happens in any gauge with spinors.

This work sheds considerably light on the gauge technique, by the explicit demonstration that the procedure respects the general relations (10), (11), etc and by the relatively simple nature of the configuration space propagators (31) and (33). Give the spectral functions (26) and (27) one may go on to construct the gauge-covariant Green functions (up to transverse parts) in x space by appropriate weighting:

$$\{S\Gamma_\mu S\}(xy,z)=\int dW\rho(W)S(x-z\mid W)\gamma_\mu S(z-y\mid W)$$

$$\{S\Gamma_{\mu\nu}S\}(xy,zw)=\int dW\rho(W)(S(x-z\mid W)\gamma_\nu S(z-w\mid W)\gamma_\mu S(w-y\mid W)$$
$$+S(x-w\mid W)\gamma_\mu S(w-z\mid W)\gamma_\nu S(z-y\mid W)).$$

† Some work is needed to pass from (32) to (33). The intermediate steps involve the use of

$$\int_0^\infty rJ_1(qr)F\left(b,b;1;\frac{-q^2}{m^2}\right)=-\Gamma^{-2}(b)G_{13}^{30}\left(\frac{1}{4}r^2m^2\,\middle|\,\begin{matrix}1\\0\;b\;b\end{matrix}\right);$$

the identifications, via the Barnes–Millin representation of G,

$$G_{13}^{30}\left(z\,\middle|\,\begin{matrix}0\\0\;b\;b\end{matrix}\right)=G_{02}^{20}(z\mid bb)=2z^bK_0(2z^{1/2})$$

$$G_{13}^{\prime 30}\left(z\,\middle|\,\begin{matrix}1\\0\;b\;b\end{matrix}\right)=-z^{-1}G_{13}^{30}\left(z\,\middle|\,\begin{matrix}0\\0\;b\;b\end{matrix}\right)$$

and the integral

$$\int z^\mu K_\nu(z)\,dz=(-i)^\mu z[(\mu-\nu-1)K_\nu(z)S_{\mu-1,\nu+1}(iz)-iK_{\nu+1}(z)S_{\mu,\nu}(iz)]$$

Naturally, these higher Green functions are no longer simple transcendental functions like (31) and (33), except in the limit of vanishing photon momentum, i.e. upon integration over photon location.

Appendix A

We wish to give what we believe is a fairly transparent proof of the invariance of $\int[\mathrm{d}A]\Delta(A)$ under transformations of Slavnov type, such as (15). The presence of space–time coordinates and indices is inessential; consider just a 'manifold' with coordinates A_i (so in practice $i=\mu, \alpha, x$) and a 'Lie group' of transformations g with coordinates g_a (in practice $a=\alpha, x$) preserving the measure $[\mathrm{d}A]$. For infinitesimal transformations we write

$$\delta A_i = D_{ia}(A)\,.\,\delta\chi_a.$$

We are interested in $\int\mu^T[\mathrm{d}A]f(A)$ for surfaces T transverse to the group action, where μ^T is the measure on T obtained from the measure $[\mathrm{d}A]$ and a left invariant measure on the group. For invariant f we know that the integral is independent of T. In particular, if T is given by the gauge condition $L_a(A)=\Lambda_a$, then the integral can be written as

$$\int\mu^T[\mathrm{d}A]f(A)=\int[\mathrm{d}A]\Delta^L(A)\delta(L(A)-\Lambda)f(A)$$

where $\Delta^L=\det F^L$ and F^L_{ab} is defined via the infinitesimal transformations of L, viz

$$\delta L = F^L\,.\,\delta\chi \qquad \text{or} \qquad F_{ab}=(\partial L_a/\partial A_i)D_{ib}.$$

Now consider A-dependent transformations of the type

$$A'=g_A(A) \qquad \text{or} \qquad A=g'_{A'}(A').$$

Given a gauge function L define L' by $L'(A')=L(A)$, and similarly $f'(A')\equiv f(A)$. Since we know that

$$\int_T\mu^T[\mathrm{d}A]f(A)=\int_{T'}\mu^{T'}[\mathrm{d}A']f'(A')$$

where T and T' are given by $L, L'=\Lambda$, we may write

$$\int[\mathrm{d}A]\Delta^L(A)\delta(L(A)-\Lambda)f(A)=\int[\mathrm{d}A']\Delta^{L'}(A')\delta(L'(A')-\Lambda)f'(A').$$

By integrating over Λ we can express this quite generally as

$$\int[\mathrm{d}A]\Delta^L(A)f(A)=\int[\mathrm{d}A']\Delta^{L'}(A')f(g'_{A'}(A'))$$

and changing variable on the right-hand side to A, we find

$$\Delta^L(A)=|\partial A'/\partial A|\Delta^{L'}(A').$$

This establishes the invariance of $\int[\mathrm{d}A]\Delta^L(A)$, provided that $\Delta^{L'}=\Delta^L$, or $F^{L'}=F^L$, which in its turn is satisfied if *L' and L differ by a gauge-invariant function.*

Consider now an infinitesimal A-dependent gauge transformation $A'=A+D(A)\,.\,\delta\chi(A)$. From the definition of F^L and L, we see that $L'(A')+$

$F^L(A) \,.\, \delta\chi(A) = L(A)$. Hence if we choose

$$\delta\chi(A) = [F^L(A)]^{-1} \,.\, \delta\Lambda$$

where $\delta\Lambda$ is gauge-invariant, we guarantee the invariance of the measure. In particular for the gauge fixing term $a \,.\, A = \Lambda$, $F = (a \,.\, D)$ and we therefore require the A dependence of the transformations to be of Slavnov type,

$$\delta A_\mu = D_\mu (a \,.\, D)^{-1} \,.\, \delta\Lambda$$

Appendix B

We wish to prove that when $a' + b' - c' = b - a$, the integral

$$\Gamma(c)\Gamma(c')I \equiv \int_0^1 \mathrm{d}t\, (1-tz)^{-a}(1-t)^{c-1}t^{c'-1}F\left(a, b; c; \frac{z(1-t)}{1-tz}\right)F(a', b'; c'; tz)$$

equals

$$I = F(a+a', b+b'; c+c'; z)/\Gamma(c+c').$$

Integrals comparable with this can be found in standard texts dealing with hypergeometric functions but none exactly having the above form. We shall therefore return to first principles for the proof.

Substituting the series expansion of F,

$$\begin{aligned} I = &\sum_{n,n'=0}^{\infty} \int_0^1 \mathrm{d}t\, (1-tz)^{-a-n}(1-t)^{c+n-1}t^{c'+n'-1}z^{n+n'} \\ &\times\left(\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(b)\Gamma(c+n)\Gamma(1+n)}\right)\left(\frac{\Gamma(a'+n')\Gamma(b'+n')}{\Gamma(a')\Gamma(b')\Gamma(c'+n')\Gamma(1+n')}\right) \\ = &\sum_{n,n'} F(a+n, c'+n'; c+c'+n+n'; z)\left(\frac{z^{n+n'}}{\Gamma(c+c'+n+n')}\right) \\ &\times\left(\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(b)\Gamma(1+n)}\right)\left(\frac{\Gamma(a'+n')\Gamma(b'+n')}{\Gamma(a')\Gamma(b')\Gamma(1+n')}\right) \\ = &\sum_r \sum_{n,n'} \left(\frac{z^r}{\Gamma(a)\Gamma(b)\Gamma(a')\Gamma(b')}\right)\left(\frac{\Gamma(a+r-n')\Gamma(c'+r-n)}{\Gamma(c+c'+r)\Gamma(1+r-n-n')}\right) \\ &\times\left(\frac{\Gamma(b+n)}{\Gamma(1+n)}\right)\left(\frac{\Gamma(a'+n')\Gamma(b'+n')}{\Gamma(c'+n')\Gamma(1+n')}\right). \end{aligned}$$

Recalling the combinatorial formula

$$\sum_n \frac{\Gamma(A+n)\Gamma(B-n)}{\Gamma(C+n)\Gamma(D-n)} = \frac{\Gamma(A+B)\Gamma(A-C+1)\Gamma(B-D+1)}{\Gamma(C+D-1)\Gamma(A+B-C-D+2)}$$

we can reduce the integral to the double sum

$$I = \sum_r \frac{z^r\Gamma(b+c'+r)}{\Gamma(a)\Gamma(a')\Gamma(b')\Gamma(c+c'+r)} \sum_{n'} \frac{\Gamma(a'+n')\Gamma(b'+n')\Gamma(a+r+n')}{\Gamma(b+c'+n')\Gamma(1+n')\Gamma(1+r-n')}.$$

Further simplification is possible because the n' summation is Saalschutzian and one can make use of the identity

$$\sum_n \frac{\Gamma(A+n)\Gamma(B+n)\Gamma(C-A+r-n-B)}{\Gamma(C+n)\Gamma(1+n)\Gamma(1+r-n)}$$

$$= \frac{\Gamma(A)\Gamma(B)\Gamma(C-A-B)\Gamma(C-A+r)\Gamma(C-B+r)}{\Gamma(C-A)\Gamma(C-B)\Gamma(C+r)\Gamma(1+r)}.$$

Recalling the condition $a'+b'-c'+a=b$, we obtain

$$I = \sum_r \frac{z^r\Gamma(a+b'+r)\Gamma(a+a'+r)}{\Gamma(a+b')\Gamma(a+a')\Gamma(c+c'+r)\Gamma(1+r)} = \frac{F(a+a', b+b'; c+c'; z)}{\Gamma(c+c')}$$

which was to be proved. As far as we know, this is a new integral identity.

Such an integral arises in the text in the formula succeeding equation (27). The substitution $y = 1+t(x-1)$ there and the identifications

$$a \to a\epsilon, \qquad b \to 1+a\epsilon, \qquad c \to 2a\epsilon,$$

$$a' \to -3\epsilon, \qquad b' \to 1-3\epsilon, \qquad c' \to -6\epsilon$$

bring it into the desired form and lead to the result (28).

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